HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2013

STUDENT NUMBER: ____________________

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 100

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 15

90 marks

Attempt Questions 11 – 16.

Start each question in a new writing booklet.

Write your student number on every writing booklet.

Question	*1-10*	*11*	*12*	*13*	*14*	*15*	*16*	*Total*
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 Let $z=3+2i$ and $w=2-3i$. What is the value of $3\overline{z}-2w$?

(A) 5

(B) −5

(C) $5+12i$

(D) $5-12i$

2 The equation $x^2+2y^2-2xy+x=8$ defines y implicitly as a function of x.

What is the value of $\frac{dy}{dx}$ at the point $(3,2)$?

(A) $\frac{1}{4}$

(B) $-\frac{1}{4}$

(C) $\frac{3}{2}$

(D) $-\frac{3}{2}$

3 Let $z=\cos\theta+i\sin\theta$. Which of the following is equal to z^3?

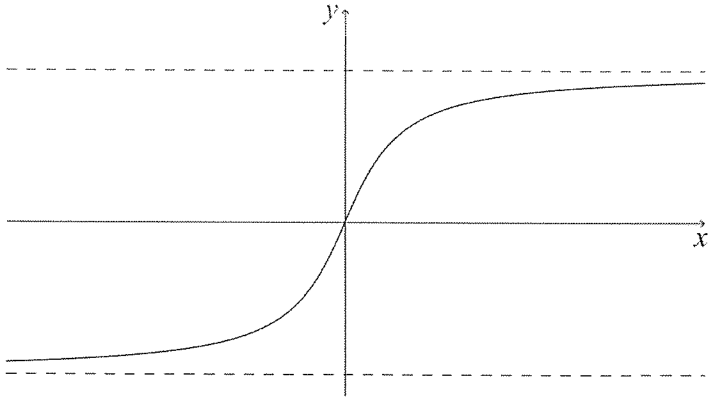
(A) $\cos^3\theta+i\sin^3\theta$

(B) $\cos^3\theta-i\sin^3\theta$

(C) $\cos 3\theta+i\sin 3\theta$

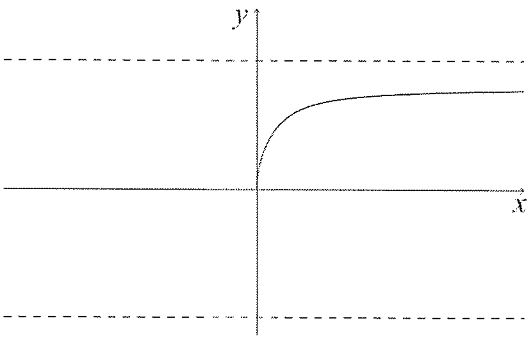
(D) $\cos 3\theta-i\sin 3\theta$

4 The graph of $y = f(x)$ is shown below.

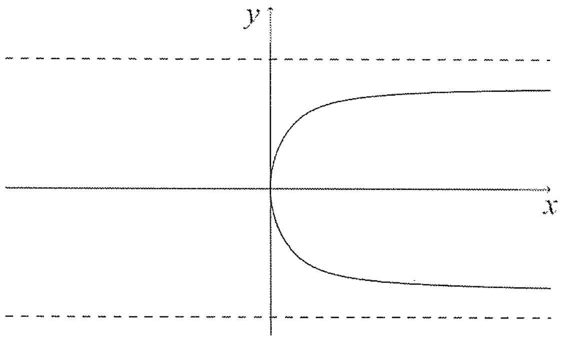


Which of the following graphs best represents $y^2 = f(x)$?

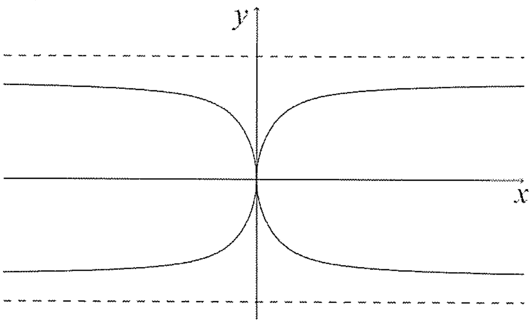
(A)



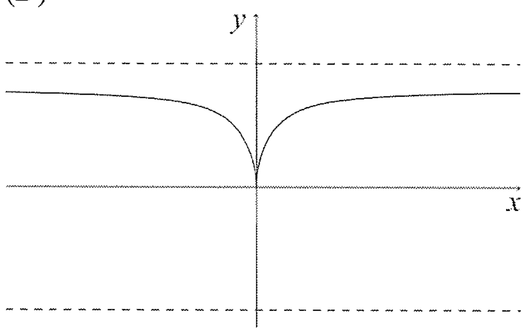
(B)



(C)



(D)



5 The roots of the polynomial $4x^3 + 4x - 5 = 0$ are α, β and γ.
What is the value of $(\alpha+\beta-3\gamma)(\beta+\gamma-3\alpha)(\alpha+\gamma-3\beta)$?

(A) -80

(B) -16

(C) 16

(D) 80

6 A mass of 5 kg moves in a horizontal circle of radius 1.5 metres at a uniform angular speed of 4 radians per second. What is the centripetal force required for this motion?

(A) 40N

(B) 80N

(C) 120N

(D) 160N

7 Which of the following is a focus of the hyperbola $\frac{x^2}{11}-\frac{y^2}{25}=-1$?

(A) $(0,5)$

(B) $(5,0)$

(C) $(6,0)$

(D) $(0,6)$

8 If $x^3-11x^2+40x-k=(x-4)^2.P(x)$, what is the value of k ?

(A) 16

(B) 32

(C) 48

(D) 64

9 The region bounded by the curve $y=x^2$, the line $x=4$ and the x-axis is rotated about the line $x=4$. Which integral represents the volume of the solid?

(A) $2\pi\int_0^4 (4-x)x^2dx$

(B) $\pi\int_0^{16} (4-x)x^2dx$

(C) $2\pi\int_0^4 (4-x)^2dx$

(D) $\pi\int_0^{16} (4-x)^2dx$

10 Without evaluating the integrals, which of the following integrals is equal to zero?

(A) $\int_{-1}^{1} e^{-x} \tan^{-1}(x^2)dx$

(B) $\int_{-1}^{1} \frac{x^2 \sin x}{x^2+5} dx$

(C) $\int_{-1}^{1} \sqrt{x^2+e^x}\, dx$

(D) $\int_{-1}^{1} x^3 \sin^{-1} x\, dx$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations

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Question 11 (15 marks) Start a new writing booklet

(a) (i) Using the substitution $x = a - u$, show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **2**

(ii) Hence evaluate $\int_0^2 x\sqrt{2-x}dx$. **2**

(b) Express $\frac{3\sqrt{3}+i}{\sqrt{3}-i}$ in the form $x+iy$, where x and y are real. **2**

(c) Find $\int e^x \cos x \, dx$. **2**

(d) Find the square roots of $1+\sqrt{3}i$. **2**

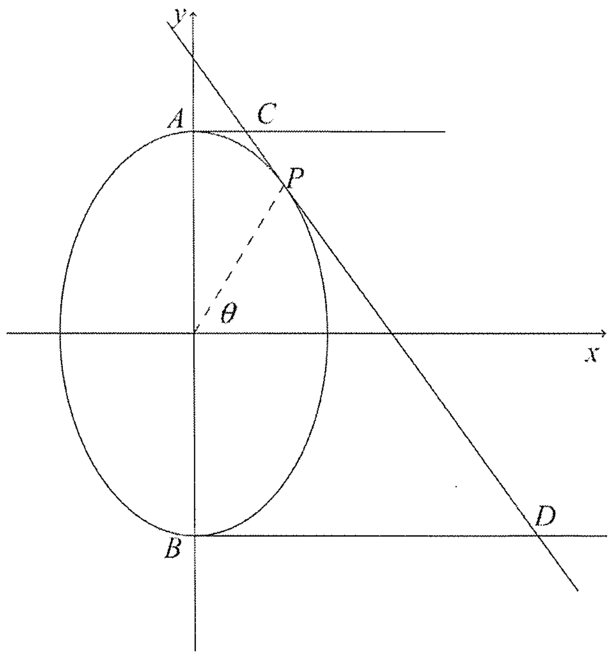
(e) Given that α, β and γ are the roots of $x^3 + px^2 + qx + r = 0$, find the equation whose roots are α^2, β^2 and γ^2. **2**

(f) Sketch the region in the complex plane where both the inequalities $|z-2-2i| < 2$ and $0 < \arg(z-2-2i) < \frac{\pi}{4}$ hold true simultaneously. **3**

Question 12 (15 marks) Start a new writing booklet

(a) Find $\int \frac{1}{8+5\sin x}\,dx$. **2**

(b) The diagram below shows the ellipse which has equation $\frac{x^2}{4}+\frac{y^2}{9}=1$. The point $P(2\cos\theta, 3\sin\theta)$, where θ is the axillary angle, lies on the ellipse. The ellipse meets the y-axis at the points A and B. The tangents to the ellipse at A and B meet the tangent at P at the points C and D respectively.



NOT TO SCALE

(i) Find the eccentricity, coordinates of the foci and the equation of the directricies. **3**

(ii) Show that the equation of the tangent to the ellipse at P is $2y\sin\theta+3x\cos\theta=6$. **2**

(iii) Find the numerical value of $AC\times BD$. **3**

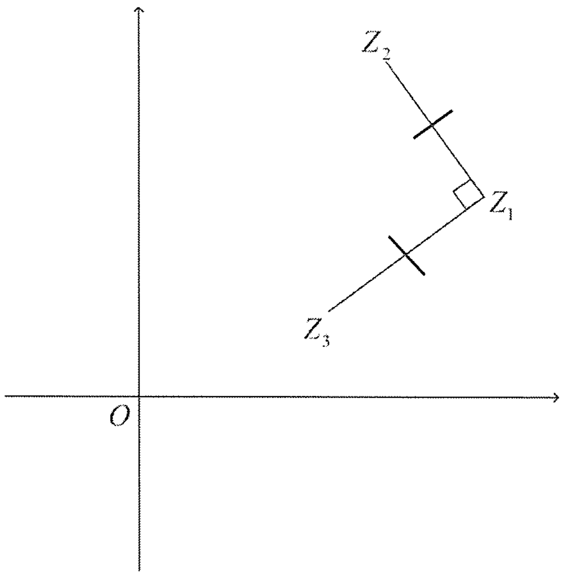
(c) For every integer $n\geq 0$, let $I_n=\int_0^{\frac{\pi}{6}}\sec^n x\,dx$. **3**

Show that for $n\geq 2$, $(n-1)I_n=\frac{2^{n-2}}{\left(\sqrt{3}\right)^{n-1}}+(n-2)I_{n-2}$.

Question 12 continues on page 9

Question 12 (continued)

(d)



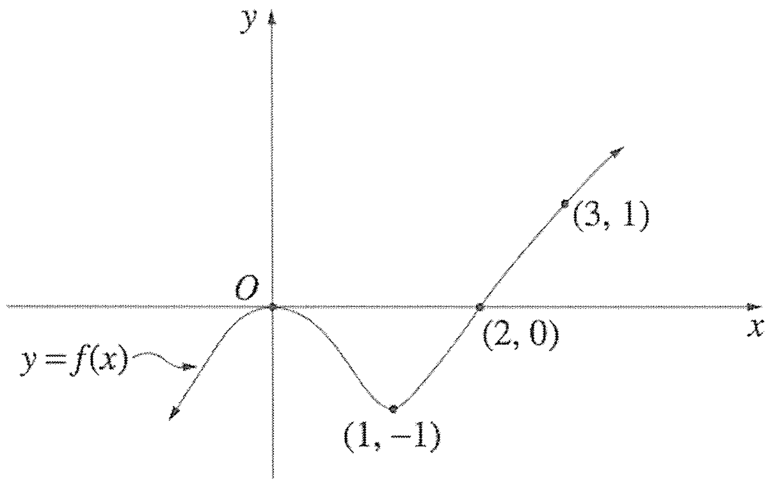
On the Argand diagram above, the point Z_1 represents the complex number z_1 and the point Z_2 represents the complex number z_2. The point Z_2 is rotated about Z_1 through a right angle in the positive direction to take up the position Z_3, representing the complex number z_3. **2**

Show that $z_3 = (1-i)z_1 + iz_2$.

End of Question 12

Question 13 (15 marks) Start a new writing booklet

(a) The diagram below shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$. **2**

(ii) $y = |f(x)|$. **2**

(iii) $y = \ln(f(x))$. **2**

(b) (i) By using De Moivre's Theorem, show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. **2**

(ii) Hence show that $\tan 3\theta = \dfrac{3t - t^3}{1 - 3t^2}$, where $t = \tan\theta$. **1**

(iii) Hence find the general solutions of the equation $3\tan\theta - \tan^3\theta = 0$ **1**

(c) (i) Find the five roots of the equation $z^5 = 1$ **2**

(ii) Show that $z^5 - 1 = (z-1)\left(z^2 - 2z\cos\dfrac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\dfrac{4\pi}{5} + 1\right)$. **2**

(iii) Hence show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$ **1**

Question 14 (15 marks) Start a new writing booklet

(a) Find $\int \frac{x+3}{x^3+x^2+x+1}\,dx$. **2**

(b) The base of a solid is the circle $x^2+y^2=36$. Find the volume of the solid if every section perpendicular to the x-axis is a square where one side of the square is completely laid in the base of the solid. **3**

(c) A parachutist of mass m falls to the ground from a plane. Air resistance is proportional to mv^2, where v is his speed and g is acceleration due to gravity. Take downwards as being the positive direction, and the point where the parachutist jumps out the plane as the origin of displacement, x.

(i) Deduce that $\frac{d}{dx}\left(v^2\right)=2g-2kv^2$, where k is the constant of proportionality. **1**

(ii) Show that $v^2=\frac{g}{k}-\frac{g}{k}e^{-2kx}$, satisfies the differential equation in part (i). **2**

(iii) Find an expression for the terminal speed of the parachutist during his free-fall. **1**

(d) Let $f(x)=3x^5-10x^3+16x$.

(i) Show that $f'(x)\geq 1$ for all real x. **2**

(ii) For what values of x is $f''(x)>0$. **2**

(iii) Sketch the graph of $y=f(x)$, clearly indicating any turning points and points of inflexion. **2**

Question 15 (15 marks) Start a new writing booklet

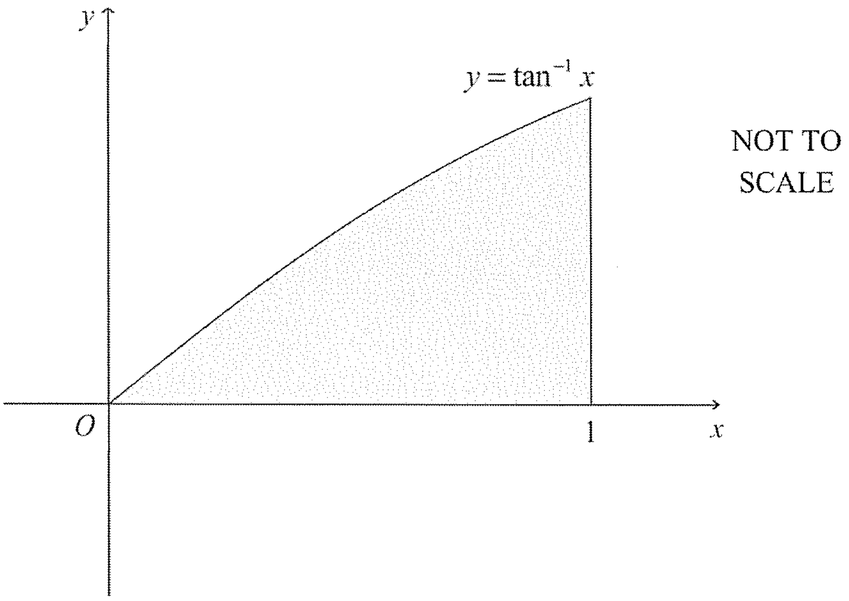
(a) Consider the function $f(u) = \sin^{-1} u - \sqrt{1-u^2}$, with restricted domain $0 < u < 1$. **1**

(i) Show that $f'(u) = \sqrt{\dfrac{1+u}{1-u}}$.

(ii) Hence, given that α is in the domain, show that **2**

$$\int_0^{\alpha} \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du = \sin^{-1}\alpha - \sqrt{1-\alpha^2} + 1$$

(b) The region bounded by the curve $y = \tan^{-1} x$ and the x axis between $x = 0$ and $x = 1$ is rotated through one complete revolution about $x = 1$. A diagram of the region to be rotated is shown below.



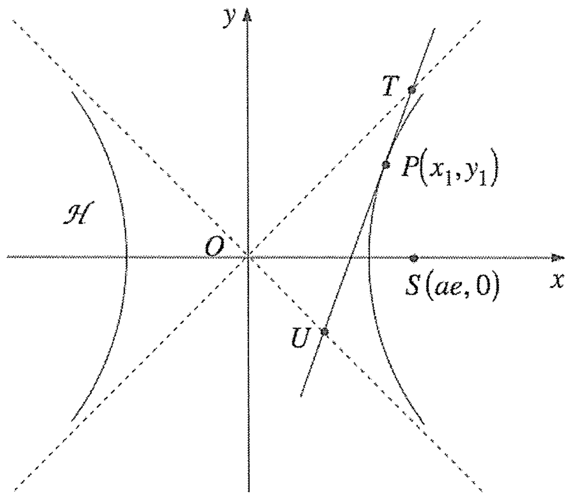
(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V = 2\pi \int_0^1 (1-x)\tan^{-1} x \, dx$. **1**

(ii) Hence find the volume V in simplest exact form. **4**

Question 15 continues on page 13

Question 15 (continued)

(c) The point $S(ae,0)$ is a focus on the hyperbola $H: x^2 - y^2 = a^2$. The tangent to the hyperbola at a point $P(x_1, y_1)$ meets the asymptotes of H at T and U, as shown in the diagram below.



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(i) Show that the equation of the tangent TU is $x_1 x - y_1 y = a^2$. **2**

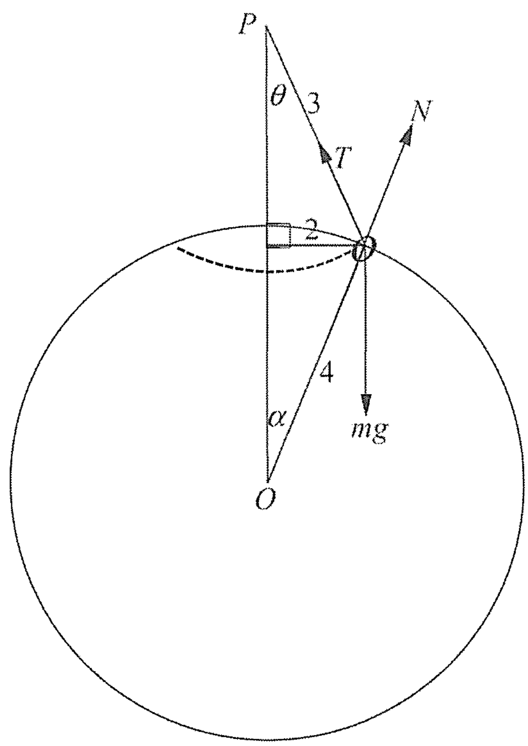
(ii) Show that the gradient of SU is $\dfrac{a}{e(x_1 + y_1) - a}$. **2**

(iii) Let $\angle UST = \theta$. Show that $\tan\theta = -1$. **3**

End of Question 15

Question 16 (15 marks) Start a new writing booklet

(a) A particle of mass 5 kg at the end of a string 3 metres long is suspended from a point P vertically above the highest point of a smooth sphere of radius 4 metres. It describes a horizontal circle of radius 2 metres on the surface of the sphere.



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Three forces act on the particle: the tension force F of the string, the normal reaction force N to the surface of the sphere, and the gravitational force mg. Take g, the acceleration due to gravity, as $10\ \text{ms}^{-2}$. The angular velocity of the particle moving in uniform circular motion is 1 radian per second.

(i) By resolving the forces horizontally and vertically on a diagram, show that **2**

$$\frac{T\sqrt{5}}{3}+\frac{N\sqrt{3}}{2}=50 \quad \text{and} \quad \frac{2T}{3}-\frac{N}{2}=10 .$$

(ii) Find, correct to one decimal place:

(α) the tension in the string. **1**

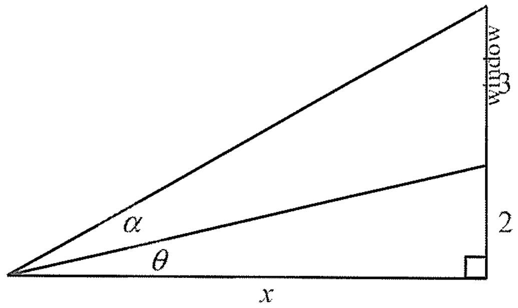
(β) The force exerted on the sphere. **1**

(iii) Find the angular velocity that will ensure there is no force exerted on the sphere. **1**

Question 16 continues on page 15

Question 16 (continued)

(b) The base of a stained glass window 3 metres high is 2 metres above the eye-level of an observer who is x metres from the base of the wall which is supporting the window. α is the viewing angle at eye level (i.e. the difference between the angles of elevation of the top and bottom of the window, as seen by the observer)



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(i) Show that $\alpha = \tan^{-1}\left(\dfrac{3x}{x^2+10}\right)$. **3**

(ii) Hence find how far should the observer stand from the wall for the viewing angle to be greatest. **3**

(c) Given that $f(x) = x^6 + 4x^5 - 3x^4 - 8x^3 + 35x^2 - 60x - 225$ has zeroes at $x = \pm\sqrt{5}$ and a double zero, factorise $f(x)$ over the:

(i) real field. **3**

(ii) complex field. **1**

End of Paper

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QUESTION 11.

a) i) let $x = a - u$ $\therefore u = a - x$.
$dx = -du$ $du = -dx$.
when $x = 0$ $u = a$
$x = a$ $u = 0$

$$\therefore \int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du$$
$$= \int_0^a f(a-u)\,du$$
$$= \int_0^a f(a-x)\,dx.$$

OR
$$\int_0^a f(a-x)\,dx = -\int_a^0 f(u)\,du$$
$$= \int_0^a f(u)\,du$$
$$= \int_0^a f(x)\,dx$$

ii)
$$\int_0^2 x\sqrt{2-x}\,dx = \int_0^2 (2-x)\sqrt{x}\,dx$$
$$= \int_0^2 \left(2x^{1/2} - x^{3/2}\right) dx$$
$$= \left[\frac{4x^{3/2}}{3} - \frac{2x^{5/2}}{5}\right]_0^2$$
$$= \frac{4\sqrt{8}}{3} - \frac{2\sqrt{32}}{5}$$
$$= \frac{16\sqrt{2}}{15}$$
$$= 1.50849$$

b)
$$\frac{3\sqrt{3}+i}{\sqrt{3}-i} = \frac{3\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i}$$
$$= 2 + i\sqrt{3}$$

MC
1. A
2. D
3. C
4. B
5. A
6. C
7. D
8. C
9. A
10. B

c) let $u = \cos x$ $dv = e^x\,dx$
$du = -\sin x\,dx$ $v = e^x$

$$\therefore \int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$$

let $u = \sin x$ $dv = e^x\,dx$
$du = \cos x\,dx$ $v = e^x$

$$\therefore \int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx.$$
$$= \frac{e^x}{2}(\cos x + \sin x)$$

d) let $(x+iy)^2 = 1 + \sqrt{3}i$
$x^2 + 2ixy - y^2 = 1 + \sqrt{3}i$
$\therefore x^2 - y^2 = 1$ ——— (1)
$2xy = \sqrt{3}$ ——— (2)
from (2) $\Rightarrow y = \frac{\sqrt{3}}{2x}$ ——— (3)

subst (3) into (1) $\Rightarrow x^2 - \frac{3}{4x^2} = 1$

$$4x^4 - 4x^2 - 3 = 0$$
$$(2x^2-3)(2x^2+1) = 0$$
$$x^2 = \frac{3}{2}, -\frac{1}{2}$$
$$\therefore x = \frac{\pm\sqrt{3}}{\sqrt{2}}$$

when $x = \frac{\sqrt{3}}{\sqrt{2}}$, $y = \frac{\sqrt{2}}{2}$

when $x = -\frac{\sqrt{3}}{\sqrt{2}}$, $y = \frac{-\sqrt{2}}{2}$

$\therefore$ sq roots are: $\pm\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{2}i}{2}\right)$

OR $\pm\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}i}{2}\right)$

e) let $y = \alpha^2$

$\therefore \quad \alpha = \sqrt{y}$

$\therefore (\sqrt{y})^3 + p(\sqrt{y})^2 + q\sqrt{y} + r = 0$

$y\sqrt{y} + py + q\sqrt{y} + r = 0$

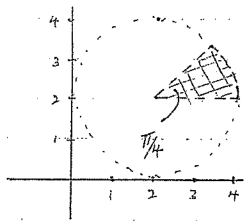
$\sqrt{y}(y+q) = -py - r$

$y(y+q)^2 = (py+r)^2$

$y^3 + 2qy^2 + q^2y = p^2y^2 + 2pry + r^2$

$\therefore y^3 + y^2(2q - p^2) + y(q^2 - 2pr) - r^2 = 0$

f)



Question 12

(a) let $t = \tan\frac{x}{2}$

$$I = \int \frac{dx}{8 + 5\sin x}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

$$\therefore I = \int \frac{1}{8 + \frac{10t}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{8 + 8t^2 + 10t}$$

$$= \int \frac{dt}{4t^2 + 5t + 4}$$

$$= \frac{1}{4}\int \frac{1}{t^2 + \frac{5}{4}t + 1}$$

$$= \frac{1}{4}\int \frac{dt}{(t+\frac{5}{8})^2 + \frac{39}{64}}$$

$$= \frac{1}{4} \times \frac{8}{\sqrt{39}} \tan^{-1}\frac{(t+\frac{5}{8})8}{\sqrt{39}} + C$$

$$= \frac{2}{\sqrt{39}} \tan^{-1}\left(\frac{8\tan\frac{x}{2} + 5}{\sqrt{39}}\right) + C$$

(B) (i) $a^2=b^2(1-e^2)$

$1-e^2=\frac{a^2}{b^2}$

$e^2=1-\frac{a^2}{b^2}$

$e^2=1-\frac{4}{9}$

$e^2=\frac{5}{9}$

$e=\frac{\sqrt{5}}{3}$

∴ Focus $(0, \pm\sqrt{5})$

Directrices $y=\pm\frac{9}{\sqrt{5}}$

(ii) $\frac{x^2}{4}+\frac{y^2}{9}=1$

$\frac{2x}{4}+\frac{2y}{9}\cdot\frac{dy}{dx}=0$

$\frac{2y}{9}\cdot\frac{dy}{dx}=-\frac{x}{2}$

$\frac{dy}{dx}=\frac{-9x}{4y}$

When $x=2\cos\theta$, $y=3\sin\theta$

$\frac{dy}{dx}=\frac{-18\cos\theta}{12\sin\theta}$

$=\frac{-3\cos\theta}{2\sin\theta}$

(iii) When $y=3$

$6\sin\theta+3x\cos\theta=6$

$3x\cos\theta=6-6\sin\theta$

$x=\frac{2-2\sin\theta}{\cos\theta}$

When $y=-3$

$-6\sin\theta+3x\cos\theta=6$

$x=\frac{2+2\sin\theta}{\cos\theta}$

∴ $AC\times BD=\frac{4-4\sin^2\theta}{\cos^2\theta}$

$=\frac{4(1-\sin^2\theta)}{\cos^2\theta}$

$=4\frac{\cos^2\theta}{\cos^2\theta}$

$=4.$

(c) $I_n=\int_0^{\pi/6}\sec^n x\,dx$

Let $u=\sec^{n-2}x$

$u'=(n-2)\sec^{n-1}x\sin x$

$v'=\sec^2 x$

$v=\tan x$

∴ $I_n=\left[\tan x\sec^{n-2}x\right]_0^{\pi/6}-(n-2)\int_0^{\pi/6}\tan x\sin x\sec^{n-1}x\,dx$

$=\frac{1}{\sqrt{3}}\cdot\left(\frac{2}{\sqrt{3}}\right)^{n-2}-(n-2)\int_0^{\pi/6}\sin^2 x\sec^n x\,dx$

$=\frac{2^{n-2}}{\sqrt{3}^{n-1}}-(n-2)\int_0^{\pi/6}(1-\cos^2 x)\sec^n x\,dx$

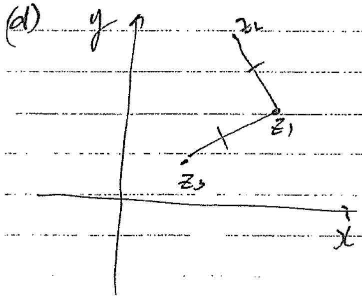
$=\frac{2^{n-2}}{\sqrt{3}^{n-1}}-(n-2)\int_0^{\pi/2}(\sec^n x\,dx-\sec^{n-2}x)\,dx$

$=\frac{2^{n-2}}{\sqrt{3}^{n-1}}-(n-2)I_n+(n-2)I_{n-2}.$

$I_n(1+n-2)=\frac{2^{n-2}}{(\sqrt{3})^{n-1}}+(n-2)I_{n-2}$

$I_n(n-1)=\frac{2^{n-2}}{(\sqrt{3})^{n-1}}+(n-2)I_{n-2}$

(d)



$\overrightarrow{z_1z_2}=z_2-z_1$

$\overrightarrow{z_1z_3}=i(z_2-z_1)$

~~$\overrightarrow{Oz_3}=\overrightarrow{Oz_1}+\overrightarrow{z_1z_3}$~~

$\overrightarrow{Oz_3}=\overrightarrow{Oz_1}+\overrightarrow{z_1z_3}$

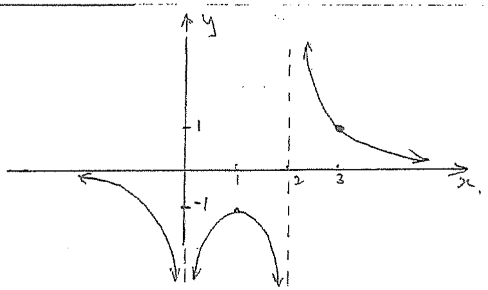
∴ $z_3=z_1+i(z_2-z_1)$

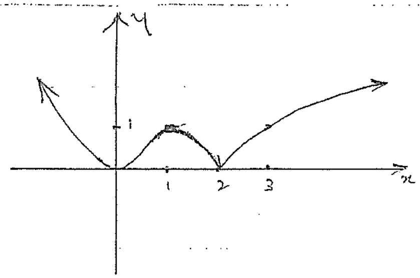
$z_3=z_1+iz_2-iz_1$

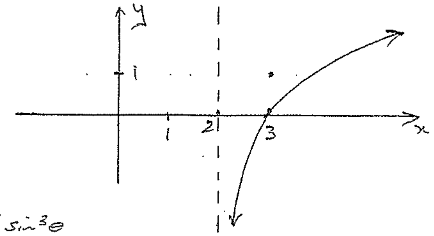
$z_3=z_1(1-i)+iz_2$

QUESTION 13.

a)







b) i) let $z = \cos\theta + i\sin\theta$

$z^3 = \cos 3\theta + i\sin 3\theta$

$z^3 = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

equating real & imaginary parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

ii) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$

$= \dfrac{\sin\theta(3\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - 3\sin^2\theta)} \div \dfrac{\cos^2\theta}{\cos^2\theta}$

$= \tan\theta\left(\dfrac{3 - \tan^2\theta}{1 - 3\tan^2\theta}\right)$

$= \dfrac{t(3 - t^2)}{1 - 3t}$ where $t = \tan\theta$

$= \dfrac{3t - t^3}{1 - 3t^2}$

iii) $3\tan\theta - \tan^3\theta = 0$

let $t = \tan\theta$

$3t - t^3 = 0$

$\dfrac{3t - t^3}{1 - 3t^2} = 0$

$\tan 3\theta = 0$

$3\theta = n\pi$

$\theta = \dfrac{n\pi}{3}$ (n is an integer)

OR $\tan\theta(3 - \tan^2\theta) = 0$

$\tan\theta = 0$ $\tan\theta = \pm\sqrt{3}$

$\theta = n\pi$ $\theta = n\pi \pm \frac{\pi}{3}$

$\therefore \theta = n\pi$

$\theta = n\pi \pm \frac{\pi}{3}$

c) i) $z_0 = \cos 0 + i\sin 0 = 1$

$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_3 = \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} = \overline{z_2}$

$z_4 = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = \overline{z_1}$

ii) $z^5 - 1 = (z - z_0)(z - z_1)(z - \overline{z_1})(z - z_2)(z - \overline{z_2})$

$= (z-1)(z^2 - z(z_1 + \overline{z_1}) + z_1\overline{z_1})(z^2 - z(z_2 + \overline{z_2}) + z_2\overline{z_2})$

now $z_1 + \overline{z_1} = 2\cos\frac{2\pi}{5}$ $z_1\overline{z_1} = \cos^2\frac{2\pi}{5} + \sin^2\frac{2\pi}{5} = 1$

$z_2 + \overline{z_2} = 2\cos\frac{4\pi}{5}$ $z_2\overline{z_2} = \cos^2\frac{4\pi}{5} + \sin^2\frac{4\pi}{5} = 1$

$\therefore z^5 - 1 = (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$

iii) sum of roots $= -\frac{b}{a} = 0 = 1 + z_1 + \overline{z_1} + z_2 + \overline{z_2}$

$0 = 1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5}$

$-1 = 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5}$

$-\frac{1}{2} = \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$

Question 14 Solutions

(a)

$$\int \frac{x+3}{x^3+x^2+x+1} = \int \frac{x+3}{x^2(x+1)+(x+1)}\,dx$$

$$= \int \frac{x+3}{(x^2+1)(x+1)}\,dx$$

Let $\frac{x+3}{(x^2+1)(x+1)} \equiv \frac{Ax+B}{(x^2+1)} + \frac{C}{x+1}$

$x+3 = (Ax+B)(x+1) + C(x^2+1)$

Let $x=-1$: $2 = 2C$, $\therefore C = 1$

$x = 0$: $3 = B+1$, $B = 2$

$x = 1$: $4 = 2A+2B+2C$, $2 = A+2+1$, $A = -1$

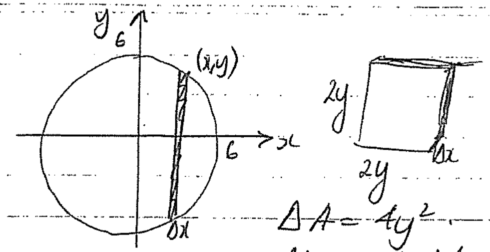
$$\therefore \int \frac{x+3}{(x^2+1)(x+1)}\,dx = \int \frac{-x+2}{x^2+1}\,dx + \int \frac{1}{x+1}\,dx$$

$$= \frac{-1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx + \int \frac{1}{x+1}\,dx$$

$$= \frac{-1}{2}\ln(x^2+1) + 2\tan^{-1}x + \ln(x+1) + C$$

$$= \ln\left(\frac{x+1}{\sqrt{x^2+1}}\right) + 2\tan^{-1}x + C$$

(b)



$\Delta A = 4y^2$

$\Delta V = 4y^2\Delta x$

$x^2+y^2 = 36$

$y^2 = 36 - x^2$

$4y^2 = 144 - 4x^2$

$$\Delta V \doteqdot \sum_{x=-6}^{6}(144-4x^2)\Delta x$$

$$= \lim_{\Delta x \to 0} \sum_{x=-6}^{6}(144-4x^2)\Delta x$$

$$= \int_{-6}^{6}(144-4x^2)\,dx$$

$$= \left[144x - \frac{4x^3}{3}\right]_{-6}^{6}$$

$$= (864-288) - (-864+288)$$

$$= 1152 \text{ units}^3.$$

(i) $F = ma$

$ma = mg - kmv^2$

$a = g - kv^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - kv^2$

$\frac{1}{2}\frac{d}{dx}(v^2) = g - kv^2$

$\frac{d(v^2)}{dx} = 2g - 2kv^2$

) $\frac{dx}{dv^2} = \frac{1}{2g - 2kv^2}$

$x = \frac{1}{2}\int \frac{1}{g - kv^2}\, d(v^2)$

$x = \frac{-1}{2k}\int \frac{-k}{g - kv^2}\, d(v^2)$

$x = \frac{-1}{2k}\ln(g - kv^2) + C$

when $x = 0,\ v = 0$

$0 = \frac{-1}{2k}\ln g + C$

$\therefore C = \frac{1}{2k}\ln g$

$x = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$

$\frac{g}{g - kv^2} = e^{2kx}$

$g - kv^2 = ge^{-2kx}$

$-kv^2 = ge^{-2kx} - g$

$v^2 = \frac{g}{k} - \frac{g}{k}e^{-2kx}$

Alternate:

$\frac{d}{dx}(v^2) = \frac{d}{dx}\left(\frac{g}{k} - \frac{g}{k}e^{-2kx}\right)$

$= -2k \times -\frac{g}{k}e^{-2kx}$

$= 2ge^{-2kx}$

$= 2g\left(e^{-2kx}\right)$

But $v^2 = \frac{g}{k} - \frac{g}{k}e^{-2kx}$

$v^2 - \frac{g}{k} = -\frac{g}{k}e^{-2kx}$

$kv^2 - g = -ge^{-2kx}$

$e^{-2kx} = 1 - \frac{kv^2}{g}$

$\therefore \frac{d}{dx}(v^2) = 2g\left(1 - \frac{kv^2}{g}\right)$

$= 2g - 2kv^2$

(iii) $a = g - kv^2$

As $a \to 0$

$0 = g - kv^2$

$kv^2 = g$

$v^2 = \frac{g}{k}$

$v = \sqrt{\frac{g}{k}}$

$\lim_{x\to\infty} v^2 = \lim_{x\to\infty} \frac{g}{k} - \frac{g}{ke^{2kx}}$

$= \frac{g}{k}$

$\therefore v \to \sqrt{\frac{g}{k}}$

(d) (i) $f(x) = 3x^5 - 10x^3 + 16x$

$f'(x) = 15x^4 - 30x^2 + 16$

$= 15(x^4 - 2x^2) + 16$

$= 15(x^4 - 2x^2 + 1) + 1$

$= 15(x^2 - 1)^2 + 1$

$(x^2 - 1)^2 \geq 0$

$15(x^2 - 1)^2 \geq 0$

$15(x^2 - 1)^2 + 1 \geq 1$

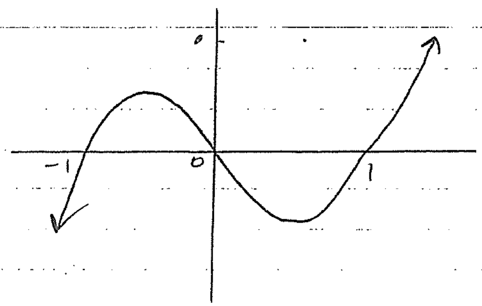
$\therefore f'(x) \geq 1$ for all x.

(ii) $f''(x) = 60x^3 - 60x$

Let $f''(x) > 0$

$60x^3 - 60x > 0$

$60x(x-1)(x+1) > 0$



$\therefore f''(x) > 0$ for $x > 1,\ -1 < x < 0$.

(iii) $f'(x) > 0$ no stationary points

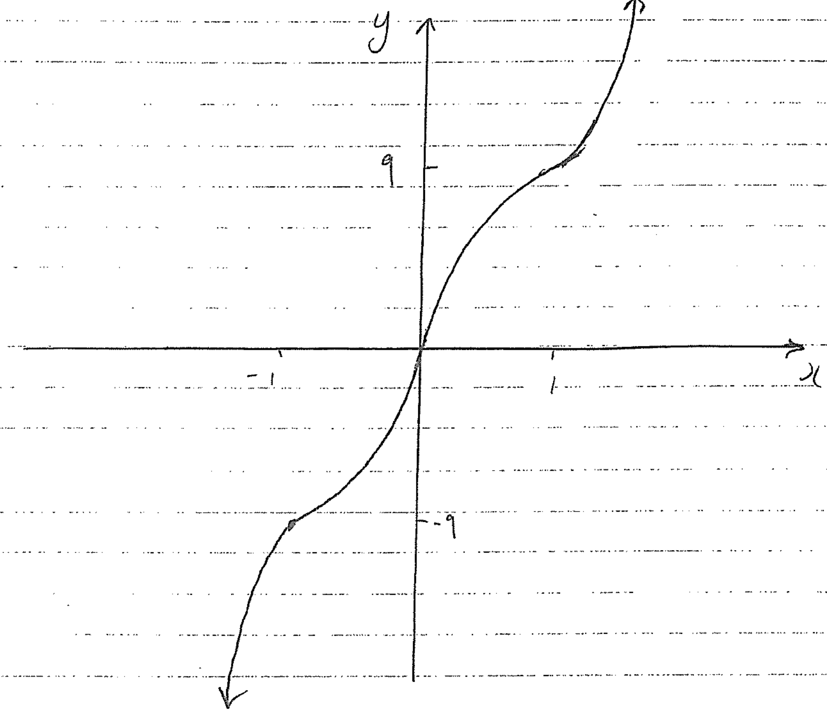
$f''(x) = 0$ for $x = -1, 0, 1$

$\therefore$ 3 pts of inflexion (changes sign (ii))

When $x = -1$, $f(-1) = -9$, $f'(-1) = 1$

$x = 0$, $f(0) = 0$, $f'(0) = 16$

$x = 1$, $f(1) = 9$, $f'(1) = 1$



QUESTION 15.

a) i) Let $y = \sin^{-1}(u) - (1-u^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} - \frac{1}{2}(1-u^2)^{-1/2} \cdot -2u$$

$$= \frac{1}{(1-u^2)^{1/2}} + \frac{u}{(1-u^2)^{1/2}}$$

$$= \frac{1+u}{\sqrt{1-u^2}}$$

$$= \frac{\sqrt{1+u} \times \sqrt{1+u}}{\sqrt{1-u} \times \sqrt{1+u}}$$

$$= \frac{\sqrt{1+u}}{\sqrt{1-u}}$$

ii) $$\int_0^{\alpha} \left(\frac{1+u}{1-u}\right)^{1/2} du = \left[\sin^{-1}u - (1-u^2)^{1/2}\right]_0^{\alpha}$$

$$= \sin^{-1}\alpha - \sqrt{1-\alpha^2} - \sin^{-1}0 + 1$$

$$= \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2}$$

b) i) $\Delta V = 2\pi(1-x)\,y\,\Delta x$

$\Delta V = 2\pi(1-x)\tan^{-1}x \, \Delta x$

$$V = \lim_{\Delta x \to 0} \sum_0^1 2\pi(1-x)\tan^{-1}x \, \Delta x$$

$$= 2\pi \int_0^1 (1-x)\tan^{-1}x \, dx.$$

ii) Let $u = \tan^{-1}x$ $\quad dv = (1-x)\,dx$

$du = \dfrac{dx}{1+x^2}$ $\quad v = x - \dfrac{x^2}{2}$

$$V = 2\pi\left[\left[\left(x - \frac{x^2}{2}\right)\tan^{-1}x\right]_0^1 - \int_0^1 \frac{2x-x^2}{2} \times \frac{1}{1+x^2}\,dx\right]$$

$$= \frac{\pi^2}{4} + \pi\int_0^1 \left(\frac{x^2+1}{x^2+1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right) dx.$$

$$= \frac{\pi^2}{4} + \pi\left[x - \ln|x^2+1| - \tan^{-1}x\right]_0^1 \quad = \pi(1 - \ln 2)$$

c) $x^2 - y^2 = a^2$ —— ①

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y}$

$\therefore y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$yy_1 - y_1^2 = xx_1 - x_1^2$

$xx_1 - yy_1 = x_1^2 - y_1^2$

$xx_1 - yy_1 = a^2$ (as $x_1^2 - y_1^2 = a^2$ from Eqn ①)

ii) $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$

$\therefore$ asymptotes $y = \pm x$

$a^2 = a^2(e^2 - 1)$

$e^2 = 2$

$e = \sqrt{2}$

U is on $y = -x$

solve $y = -x$ —— ①

$xx_1 - yy_1 = a^2$ —— ②

subst ① into ②

$xx_1 + xy_1 = a^2$

$x = \frac{a^2}{x_1 + y_1}$

$\therefore y = \frac{-a^2}{x_1 + y_1}$

grad SU $= \dfrac{0 + \frac{a^2}{x_1+y_1}}{ae - \frac{a^2}{x_1+y_1}}$

$= \frac{a^2}{x_1+y_1} \times \frac{x_1+y_1}{ae(x_1+y_1) - a^2}$

$= \frac{a^2}{a(e(x_1+y_1) - a)}$

$= \frac{a}{e(x_1+y_1) - a}$

iii) T is on $y = x$

$\therefore$ solve $y = x$ —— ①

$xx_1 - yy_1 = a^2$ —— ②

subst ① into ② $\Rightarrow xx_1 - xy_1 = a^2$

$x = \frac{a^2}{x_1 - y_1}$

$\therefore y = \frac{a^2}{x_1 - y_1}$

grad ST $= \dfrac{\frac{-a^2}{x_1-y_1}}{ae - \frac{a^2}{x_1-y_1}}$

$= \frac{-a^2}{x_1-y_1} \times \frac{x_1-y_1}{ae(x_1-y_1) - a^2}$

$= \frac{-a^2}{a(e(x_1-y_1) - a)}$

$= \frac{a}{a - e(x_1-y_1)}$

$\tan\theta = \left| \dfrac{\frac{a}{e(x_1+y_1)-a} - \frac{a}{a-e(x_1-y_1)}}{1 + \frac{a^2}{(e(x_1+y_1)-a)(a-e(x_1-y_1))}} \right|$

$= \left| \frac{a^2 - ae(x_1-y_1) - ae(x_1+y_1) + a^2}{(e(x_1+y_1) - a)(a - e(x_1-y_1)) + a^2} \right|$

$= \left| \frac{2a^2 - aex_1 + aey_1 - aex_1 - aey_1}{ae(x_1+y_1) - e^2(x_1^2 - y_1^2) - a^2 + ae(x_1-y_1) + a^2} \right|$

$= \left| \frac{2a^2 - 2aex_1}{ae(x_1+y_1+x_1-y_1) - a^2e^2} \right|$ (as $x_1^2 - y_1^2 = a^2$)

$= \left| \frac{2a(a - ex_1)}{2aex_1 - a^2e^2} \right|$

now $e^2 = 2$.

$= \left| \frac{2a(a - ex_1)}{2aex_1 - 2a^2} \right|$

$= \left| \frac{2a(a - ex_1)}{2a(ex_1 - a)} \right|$

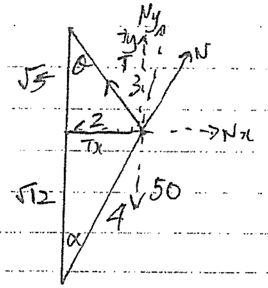
$= |-1|$

$= 1$

$\therefore \tan\theta = -1$ as θ is obtuse.

16.

(a) (i)



$\sin\theta = \frac{2}{3}$ $\cos\theta = \frac{\sqrt{5}}{3}$

$\sin\alpha = \frac{1}{2}$ $\cos\alpha = \frac{\sqrt{3}}{2}$

vertically:

$T\cos\theta + N\cos\alpha = 50$ ... ①

~~Horizontally~~:

$T\sin\theta - N\sin\alpha = m\omega^2 r$

$= 10$ ... ②

In ①

$T \times \frac{\sqrt{5}}{3} + N \cdot \frac{\sqrt{3}}{2} = 50$

$\frac{\sqrt{5}T}{3} + \frac{\sqrt{3}N}{2} = 50$

In ②:

$T \times \frac{2}{3} - N \times \frac{1}{2} = 10$

$\frac{2T}{3} - \frac{N}{2} = 10$

(ii) $-\frac{N}{2} = 10 - \frac{2T}{3}$.

$N = \frac{4T}{3} - 20$

$\frac{T\sqrt{5}}{3} + \left(\frac{4T}{3} - 20\right)\frac{\sqrt{3}}{2} = 50$

$T = \frac{150 + 30\sqrt{3}}{\sqrt{5} + 2\sqrt{3}}$

$T \doteq 35.43$ N.

$\therefore N = \frac{4 \times 35.43}{3} - 20$

$= 27.25$ N

~~N=0~~

(ii) $T\cos\theta = 50$

$T\sin\theta = 10\omega^2$.

$\therefore \tan\theta = \frac{10\omega^2}{50}$

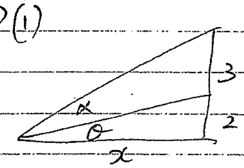
$5\tan\theta = \omega^2$

$\omega = \sqrt{5\tan\theta}$

$= \sqrt{5 \times \frac{2}{\sqrt{5}}}$

$= 2.11$ rad/s.

(b) (i)



$\tan\theta = \frac{2}{x}$

$\tan(\theta + \alpha) = \frac{5}{x}$

$\frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{5}{x}$

$\frac{2}{x} + \tan\alpha = \frac{5}{x}\left(1 - \frac{2\tan\alpha}{x}\right)$

$2 + x\tan\alpha = 5 - \frac{10\tan\alpha}{x}$

$2x + x^2\tan\alpha = 5x - 10\tan\alpha$

$x^2\tan\alpha + 10\tan\alpha = 3x$

$\tan\alpha(x^2 + 10) = 3x$

$\tan\alpha = \frac{3x}{x^2+10}$

$\alpha = \tan^{-1}\left(\frac{3x}{x^2+10}\right)$

(ii) Let $y = \frac{3x}{x^2+10}$

$$\frac{dy}{dx} = \frac{3(x^2+10) - 3x(2x)}{(x^2+10)^2}$$

$$= \frac{3x^2+30-6x^2}{(x^2+10)^2}$$

$$= \frac{30-3x^2}{(x^2+10)^2}$$

$$\frac{d\alpha}{dx} = \frac{30-3x^2}{(x^2+10)^2} \div \left(1 + \frac{9x^2}{(x^2+10)^2}\right)$$

$$= \frac{30-3x^2}{(x^2+10)^2} \times \frac{(x^2+10)^2}{(x^2+10)^2+9x^2}$$

$$= \frac{30-3x^2}{(x^2+10)^2+9x^2}$$

Let $\frac{d\alpha}{dx} = 0$

$\therefore$ $30 - 3x^2 = 0$

$x^2 = 10$

$x = \sqrt{10}$ $(x>0)$

$\therefore$

x	3	$\sqrt{10}$	4
$\frac{d\alpha}{dx}$	$+6.8\times10^{-3}$	0	-0.021

$\therefore$ Max at $x = \sqrt{10}$ m away from wall

(c) $f(x) = x^6 + 4x^5 - 3x^4 - 8x^3 + 35x^2 - 60x - 225$

$f'(x) = 6x^5 + 20x^4 - 12x^3 - 24x^2 + 70x - 60$

$(x-\sqrt{5})(x+\sqrt{5}) = x^2 - 5$

By long division:

$x^4 + 4x^3 + 2x^2 + 12x + 45$

$x^2 - 5 \,)\, x^6 + 4x^5 - 3x^4 - 8x^3 + 35x^2 - 60x - 225$

$x^6 - 5x^4$

$4x^5 + 2x^4 - 8x^3$

$4x^5 - 20x^3$

$2x^4 + 12x^3 + 35x^2$

$2x^4 - 10x^2$

$12x^3 + 45x^2 - 60x$

$12x^3 - 60x$

$45x^2 - 225$

$45x^2 - 225$

0

$\therefore f(x) = (x^2-5)(x^4+4x^3+2x^2+12x+45)$

Let $g(x) = x^4 + 4x^3 + 2x^2 + 12x + 45$

$g'(x) = 4x^3 + 12x^2 + 4x + 12$

$= 4(x^3 + 3x^2 + x + 3)$

$= 4(x^2(x+3) + 1(x+3))$

$= 4(x^2+1)(x+3)$

$\therefore$ Double root at $x = -3$

$x^2 - 2x + 5$

$x^2 + 6x + 9 \,)\, x^4 + 4x^3 + 2x^2 + 12x + 45$

$x^4 + 6x^3 + 9x^2$

$-2x^3 - 7x^2 + 12x$

$-2x^3 - 12x^2 - 18x$

$5x^2 + 30x + 45$

$5x^2 + 30x + 45$

0

$\Delta = 4 - 20 < 0$

$\therefore \quad f(x) = (x^2 - 5)(x^2 - 2x + 5)(x^2 + 6x + 9)$

$= (x - \sqrt{5})(x + \sqrt{5})(x + 3)^2(x^2 - 2x + 5)$

(ii)

$f(x) = (x - \sqrt{5})(x + \sqrt{5})(x + 3)^2(x^2 - 2x + 1 + 4)$

$= (x - \sqrt{5})(x + \sqrt{5})(x + 3)^2((x - 1)^2 + 4)$

$= (x - \sqrt{5})(x + \sqrt{5})(x + 3)^2(x - 1 - 2i)(x - 1 + 2i)$

$= (x - \sqrt{5})(x + \sqrt{5})(x + 3)^2(x - (1 + 2i))(x - (1 - 2i))$